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Abstract. Using a recent result of Mañé [**Ma**] we give a classification of polynomials whose Fatou components are John domains.

1. Introduction

Let P be a polynomial of degree $d \geq 2$. We denote by K the filled-in Julia set, by $J = \partial K$ the Julia set, and by $A_\infty = \mathbb{C} \backslash K$ the basin at infinity. We also denote by $\{\mathcal{F}_j\}$ the bounded Fatou components, i.e. the connected components of $K \backslash J$. For $x \in J$ we denote by $\omega(x)$ the closure of the forward orbit $\{P^n(x) : n \geq 1\}$. The purpose of this paper is to give a description of when A_∞ and the $\mathcal{F}_j$ are John domains.

Recall that a domain Ω is an (ε) John domain if there is a "center" $z_0 \in \Omega$ ($z_0 = \infty$ when Ω is unbounded and $\partial\Omega$ is compact) such that for all $z_1 \in \Omega$ there exists an arc $\gamma \subset \Omega$ connecting z_1 to z_0 and

$$\delta(z) \geq \varepsilon |z - z_1| \ , \ z \in \gamma.$$

By $\delta(z)$ we mean distance $\delta(z, \partial\Omega)$ (in the Euclidean metric). See [**C,J**], [**J**] and [**N,V**]for background on John domains.

We will prove that the property of A_∞ and all $\mathcal{F}_j$ being John domains is equivalent to a weak version of hyperbolicity of P on J. The polynomial P is said to be hyperbolic on a set E if there are $c, \eta > 0$ such that

$$\left|\frac{d}{dz} P^n(z)\right| \geq c(1 + \eta)^n \ , \ z \in E.$$

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In [**C,J**] it is proven that A_∞ and all $\mathcal{F}_j$ are John domains if P is hyperbolic on J, or if P satisfies a Thurston-Misiurewicz condition. In the latter case P is not hyperbolic on J, but it is hyperbolic with respect to a different (non-Euclidean) metric that was introduced earlier by Douady and Hubbard.

The main point of this paper is that A_∞ and all the $\mathcal{F}_j$ are John domains if and only if a certain analogue of hyperbolicity holds. We call this property semi-hyperbolicity. Roughly speaking, a polynomial is semi-hyperbolic if for any sufficiently small disk B centered on J, any choice of P^{-n} (restricted to B) has bounded ramification degree. A definition in terms of forward iterates is also given.

Theorem 1.1. *The following conditions are equivalent:*

(1.1) *P is semi-hyperbolic.*

(1.2) *P has no parabolic orbits and $c \notin \omega(c)$ whenever $P'(c) = 0$ and $c \in J$.*

(1.3) *A_∞ is a John domain.*

(1.4) *A_∞ is a John domain and every $\mathcal{F}_j$ has $\partial\mathcal{F}_j$ a quasicircle (hence $\mathcal{F}_j$ is a John domain).*

The theorem allows us to state in a very precise manner when the Julia set J is a fractal. Let us take for our definition of "fractal" that for all $x \in J$ and $r <$ diameter (J) there is a holomorphic *scaling* function $F = F_{x,r}$ defined on $B(x,2r) = \{|z-x| < 2r\}$ such that $|F(z)| \leq C$ on $B(x,2r)$,

$$F(J \cap B(x,r)) = J$$

and

$$F \text{ has bounded degree } \leq D_0 \text{ on } B(x,2r).$$

Then by classical distortion theorems, e.g. our Lemma 2.2, F behaves "like a univalent function", so that F is a bounded degree version of a *linear* scaling function. (This means we could also define J to be a fractal if there were scaling functions $F_{x,r}$ that were not necessarily holomorphic, but simply satisfied the *geometric* distortion properties of holomorphic scaling functions.) Our Proposition 2.1, Part (C') states that P is semi-hyperbolic if and only if P^n $(n = n(x,r))$ acts as a scaling function. By Theorem 1.1 we thus have

$$J \text{ is fractal } \Leftrightarrow P \text{ is semi-hyperbolic } \Leftrightarrow A_\infty \text{ is John.}$$

Theorem 1.1 also allows an almost complete classification of when various components of $\bar{\mathbb{C}}\setminus J$ are John domains, and this is most attractive when $P(z) = z^2 + c$. In that case the classification theorem of Sullivan states that either P has an attractive basin, a flower (from a parabolic periodic point), a Siegel disk, or J is a dendrite (and there are no $\mathcal{F}_j$'s). If there is a Siegel disk with center z_0 and multiplier $\lambda = e^{2\pi i\alpha}$ we have the usual number theoretic conditions α may satisfy. We say α is of constant type ($\alpha \in C.T.$) if $|\alpha - p/q| \geq cq^{-2}$ for all $p, q \in \mathbb{Z}$ and we say α satisfies a Diophantine condition ($\alpha \in D$) if $|\alpha - p/q| \geq cq^{-\tau}$ for some $c > 0$, $\tau < \infty$. We say $\alpha \in B$ if α satisfies the Brjuno condition. By combining our result with work of Michael Herman [**H**], we obtain the following table for $z^2 + c$. A slash in a column for the $\mathcal{F}_j$ indicates there are no $\mathcal{F}_j$.

Classification	A_∞ John?	All $\mathcal{F}_j$ John?	$\partial\mathcal{F}_j =$ quasicircle?
$c \notin M$	Yes	$--$	$--$
Attractive Basin	Yes	Yes	Yes
Parabolic Basin, one Petal	No	Yes	No
Parabolic Basin, ≥ 2 Petals	No	Yes	Yes
Siegel Disk, $\alpha \in C.T.$	No	Yes	Yes
Siegel Disk, $\alpha \in D \setminus C.T.$	No	?	No
Siegel Disk $\alpha \in B \setminus D$	No	?	?
Dendrite, $o \notin \omega(0)$	Yes	$--$	$--$
Dendrite, $o \in \omega(0)$	No	$--$	$--$

We remark that Michael Herman [**H**] has produced an example where $\alpha \in B \setminus D$ and where all $\partial\mathcal{F}_j$ are quasicircles.

The John condition turns out to be very closely related to hyperbolicity on the ω-critical orbit, i.e. the union of all $\omega(c), P'(c) = 0$. This in turn is closely related to the so-called Misiurewicz condition [**M**], $c \notin \omega(c)$ for all c (or some similar condition). (The Thurston condition is the special and important subcase where critical points are preperiodic but not periodic.) A recent paper of Mañé [**Ma**] shows that when condition (1.2) holds, P is semi-hyperbolic on the critical orbit, and this easily implies semi-hyperbolicity.

In certain situations one could have the $\mathcal{F}_j$ being John domains, but A_∞ is not a John domain. As the table indicates, we will show this to be the case for polynomials having a parabolic basin plus a natural side condition. (In the case of quadratic polynomials, this side condition holds automatically.) In this case there is not hyperbolicity or even a weaker semi-hyperbolicity on the critical orbit, but there is still some "vestige" of hyperbolicity, as was discovered by Douady and Hubbard [**DH**]. In this case we prove all $\mathcal{F}_j$ are John domains, and $\partial\mathcal{F}_j$ is a quasicircle if and only if there are two or more flower petals.

Section 2 is devoted to a discussion of semi-hyperbolicity and other notions of hyperbolicity. We also discuss Mañé's theorem there. These results are used in Section 3 to start the proof of Theorem 1.1, Sections 4 and 5 are devoted to the remainder of the proof of that result. Parabolic fixpoints are discussed in Section 6. In Section 7 we present the proofs for our table on "Johnness." We conclude in Section 8 with an example, $P = z^2 + c, c \in \mathbb{R}$, where one can construct J "by hand." This allows one to see more clearly how the Misiurewicz condition relates to the John geometry of A_∞. In Appendix 1 we present a technical computation for flower petals.

The work contained herein was carried out in several stages. Certain portions, e.g. the results on flower petals had been proved in 1990, and have been announced earlier. The three authors first got together as an ensemble in July 1992 at a conference on dynamical systems held at the Institut Mittag-Leffler. It was here that we discovered the notion of semi-hyperbolicity and proved Theorem 1.1. During the writing of this paper we became aware of the result of Mañé [**Ma**], which we use in our Section 2. The authors are grateful to the Institut Mittag-Leffler for the use of its facilities, and to NUTEK, the Swedish fund for Research and Development, which provided financing for the conference. The second author also wishes to thank the Göran Gustafsson Foundation for financing a 1990 visit to Stockholm, where some initial work on flower petals was done. We wish to thank Michael Herman for allowing us access to preliminary versions of his paper [**H**]. Finally, we thank Alexander Volberg for his comments which led us to rewrite Section 5.

2. Semi-hyperbolicity

For $x \in \mathbb{C}$, let $B(x,\rho)$ be the open ball $\{z : |z-x| < \rho\}$. Let $P^{-n}(z)$ be any inverse branch of P and let $B_n(x,\varepsilon)$ be a connected component of $P^{-n}(B(x,\varepsilon))$. By the maximum principle, $B_n(x,\varepsilon)$ is simply connected. P^n defines a ramified covering of B_n to B and we denote by $d_n(B_n)$ its degree.

Definition. We say that P is semi-hyperbolic if there is $\varepsilon > 0$ and $D < \infty$ so that for all $x \in J$ and all choices of inverse branches,

$$d_n(B_n(x,\varepsilon)) \leq D.$$

Remark. An equivalent definition, as is easily proved via a compactness argument, is that for all x, there exists $\varepsilon > 0$ such that $\sup_n d_n(B_n(x,\varepsilon)) < \infty$.

Theorem 2.1. *The following are equivalent:*

(A) *P is semi-hyperbolic.*

(B) *There exists $\varepsilon > 0, c > 0, 0 < \theta < 1$, and $D < \infty$ such that for $x \in J$ and $n \in \mathbb{N}$,*

$$d_n(B_n(x,\varepsilon)) \leq D,$$
$$\text{diameter}(B_n(x,\varepsilon)) \leq c\theta^n.$$

(C) *If $x \in J$ and n is the first integer such that $P^n(J \cap B(x,r)) = J$, then P^n has degree $\leq D_0$ on $B(x,2r)$, where D_0 is independent of x and r.*

(D) *P has no parabolic periodic points and for all c with $P'(c) = 0$ and $c \in J$, it holds that $c \notin \omega(c)$.*

In condition (C) we mean by "degree $\leq D_0$" that P^n is at most D_0 valent. In the proof, we shall need the following (well known) p-valent version of classical distortion theorems.

Lemma 2.2. *Let $\mathcal{D} \subset \mathbb{C}$ be a simply connected domain and let $F : \mathcal{D} \to \mathbb{D} = \{|z| < 1\}, F(\partial\mathcal{D}) \subset \partial\mathbb{D}$, be p-valent (i.e. degree p). Then if ρ denotes the hyperbolic metric,*

$$\{w \in \mathbb{D} : \rho_{\mathbb{D}}(F(z_0), w) \leq C^{-1}\} \subset F(\{z \in \mathcal{D} : \rho_{\mathcal{D}}(z, z_0) \leq 1\}) \\ \subset \{w \in \mathbb{D} : \rho_{\mathbb{D}}(F(z_0), w) \leq 1\},$$

where C depends only on p.

To prove the lemma, use Möbius transformations to reduce to the case where $z_0 = F(z_0) = 0$ and distance $(0, \partial\mathcal{D}) = 1$. Let $G(\cdot, \zeta)$ denote Green's function with pole at ζ. Then

$$-\log|F(z)| = \Sigma G_{\mathcal{D}}(z, z_j)$$

where the sum has at most p terms (and these are the points where $F(z_j) = 0$. The classical Koebe type distortion theorems applied to each term in the above sum imply there is $\frac{1}{4} < r < \frac{1}{2}$ such that

$$|F(z)| \geq C^{-1}, \quad |z| = r.$$

By Rouché, that yields the first inclusion in the statement of the lemma. The second inclusion is Schwarz's Lemma.

The main difficulty of the proof is (D) $\Rightarrow$ (A). This result is due to Mañé. As explained in the introduction, we originally had this only in the real quadratic case when results by Misiurewicz on 1-dimensional dynamics could be used. Since Mañé's theorem in our situation admits a very short proof, we include this here.

Before we begin the proof, let us note that under the conditions of theorem 2.1, P has no neutral periodic points. $B(x, \varepsilon)$ around a periodic fixed point x in f clearly violates (B). The same is true if x is in the boundary of a Siegel disk.

1. (A) $\Rightarrow$ (B). Let ε be the number in the definition of semi-hyperbolicity. We first prove that

$$\lim_{n\to\infty} \sup_{x\in J} \operatorname{diam}(B_n(x, \frac{\varepsilon}{q})) = 0.$$

Suppose this is false. We can then choose $x = x_n$ and $D_n = P^{-n}(B_n(x_n, \frac{\varepsilon}{q}))$ so that D_n tends to a simply connected domain D. This is an immediate consequence of Lemma 2.2. From Lemma 2.2 we also have $P^n(D) \subset B(x,\varepsilon)$. But any open set U in J satisfies $J \subset P^n(U)$ for all large n, which is a contradiction.

Now, choose n_0 so that whenever $n \geq n_0$

$$\operatorname{diam}\ B_n(x, \frac{\varepsilon}{2}) < \frac{\varepsilon}{4}.$$

By lemma 2.2 for some $\theta, 0 < \theta < 1$,

$$\operatorname{diam}\ B_{2n_0}(x, \frac{\varepsilon}{2}) < \frac{\varepsilon}{4}\theta$$

and we find

$$\operatorname{diam}\ B_{kn_0}(x, \frac{\varepsilon}{2}) \leq \frac{\varepsilon}{4}\theta^k.$$

2. (B) $\Rightarrow$ (D). As observed above, it follows from (B) that P has no parabolic fixed points. Suppose that for some critical point $c \in J$, $c \in \omega(c)$. We consider $B_{n_j}(c,\varepsilon)$ for a sequence n_j constructed inductively so that $c \in B_{n_j}$. B_{n_j} contains $P^{m_{j+1}}(c)$ for some suitable large m_{j+1} and we need only define $n_{j+1} = n_j + m_{j+1}$ to see that the branching from $B(c,\varepsilon)$ is unbounded which contradicts the assumption (B).

3. (B)$\Leftrightarrow$ (C). To see that (B)$\Rightarrow$ (C) let $x \in J$ and let n be the largest integer such that diameter $(P^n(B(x,r)) \leq \varepsilon/2$. Then by (B), P^n has degree $\leq D_0$ on $B(x,r)$, and by Lemma 2.2, $B(P^n(x), c\varepsilon) \subset P^n(B(x,r))$ for some $c > 0$. We now use the fact that there is an integer m such that $J \subset P^m(B(y, c\varepsilon))$ for all $y \in J$. To see that (C) $\Rightarrow$ (B), notice that Lemma 2.2 shows that (C) $\Rightarrow$ (A).

4. (D) $\Rightarrow$ (A) (Mañé). Following Mañé, we first introduce some notation. Let $S(x,\varepsilon)$ denote the square of side length 2ε and center x having sides parallel to the coordinate axis. We fix a constant $\rho > 0$ suitably small, independent of ε and x. We call $S(x,\varepsilon)$ admissible if $S(x,3\varepsilon)$ is contained in a ρ-neighborhood of J. Recall that there are no parabolic periodic points.

Lemma 2.3. *Given* $\varepsilon > 0$ *and* $N < \infty$ *there exists* $\delta > 0$ *so that if* $S(x, \delta)$ *is an admissible square and* S_n *is a connected component of* $P^{-n}(S)$ *such that the degree of* P^n *on* S_n *is* $\leq N$, *then we have for the same branch*

$$\operatorname{diam}(P^{-n}(S(x, \frac{1}{2}\delta)) \leq \varepsilon.$$

Proof. Suppose not. Then there exist $x = x_i$, $S^i = S(x_i, 2^{-i})$ admissible and diam $V_i = P^{-n_i}(S(x_i, 2^{-i-1})) \geqq \varepsilon_0 > 0$, and P^{n_i} has degree $\leq N$ on $P^{-n_i}(S^i)$. We distinguish two cases.

(i). Suppose for a sequence $i = i_\nu \to \infty$ that V_i contains a disk D_i of a positive radius r. We may assume $D_i \to D$ and $D \cap J = \phi$. Hence D is included in some Fatou component F of P. There is however no possible component. F cannot be an attractive component; there are no parabolic components. Finally, F cannot be a Siegel disk. The sequence x_i must then tend to its boundary and D must intersect the component at ∞.

Hence this case does not occur and we must have

(ii). The maximal disk in V_i has radius $\to 0$. This means, by Lemma 2.2, that $P^{-n_i}(z) \to$ constant on $S(x_i, k2^{-i}), k < 1$. But this contradicts the existence of ε_0.

We now now prove (D) $\Rightarrow$ (A).

Let δ be the number from the lemma with $N = 2^{d-1}, d =$ degree P. We shall first prove that the conclusion of lemma 2.3 holds without the assumption on the degree.

Let n be the smallest integer for which there exists $S(x, \eta), \eta \leq \delta$, admissible with $\operatorname{diam}(V) > \varepsilon, V = P^{-n}(S(x, \frac{1}{2}\eta))$. Let $V' = P^{-n}(S(x, \eta))$. Then the degree of P^n on V' is $> N$ and there exist x_1 and $x_1' \in V'$ with the following properties. For some critical point c, $P^m(x_1) = P^{m'}(x_1') = c$ and $0 < m' < m \leq n$. We take m maximal with this property. Consider

$$W' = P^m(V').$$

Then $c \in W'$ and also $P^{m-m'}(c) \in W'$. Hence for ρ suitable small, it follows that $\text{diam}(W') \geq 20N\rho$.

Let us now consider the 20 natural squares of side $\frac{1}{2}\eta$ adjacent to $S(x,\eta)$. For at least one of these, $S_1 = S(y, \frac{1}{4}\eta)$ say,

$$\text{diam}(P^{-n+m}(S_1)) \geq \rho$$

$S(y, \frac{1}{2}\eta)$ is admissible and $n - m < n$ which is a contradiction.

We have therefore proved that given $\varepsilon > 0$ for σ small enough

$$\text{diam } P^{-n}(S(x,\sigma)) \leq \varepsilon, \qquad n \geq 0.$$

The argument above now shows that the degree of P^n is $\leq 2^{d-1}$.

Remark 1. While it is true that for second degree polynomials, semi-hyperbolicity is equivalent to hyperbolicity on $\omega(c)$, this is not true even for third degree real polynomials. More precisely, there is a semi-hyperbolic real third degree polynomial such that P is not hyperbolic on both of the limit sets $\omega(c_1), \omega(c_2)$. To see this pick $c_1, c_2 \in \mathbb{R}$ such that

$$c_2 < P^3(c_1) = P^2(c_1) < c_1 < P(c_1)$$

and

$$P^2(c_2) \in [P^2(c_1), P(c_1)] \text{ and has dense orbit in this interval.}$$

Then one checks that P is semi-hyperbolic but $\omega(c_2)$ contains c_1 and hence is not hyperbolic.

Remark 2. When P is semi-hyperbolic but not hyperbolic, it is relatively easy to write down metrics λ for which P is hyperbolic. Suppose for example $P(z) = z^2 + c$ is semi-hyperbolic and $0 \in J$. Then since P is hyperbolic on $\omega(0)$, the metric $\lambda(z)|dz|$ is hyperbolic for P, where

$$\lambda(z) = \text{ distance } (z, w(0))^{-\frac{1}{2}}.$$

This is well known (see [**C,G**] or [**D,H**]) when P satisfies a Thurston condition. The λ metric is well behaved, e.g.

$$d_\lambda(z,w) \leq c|z-w|^{\frac{1}{2}}.$$

In the general case an analogous explicit metric can be written.

3. Semi-hyperbolic implies John

In this section we prove, following **[C,J]**, that (1.1) implies (1.4). That (1.3) implies (1.1) will be proved in sections 4 and 5.

Case I. J is connected. Our first step will be to prove that (1.1) implies (1.3). For this, we will require an equivalent definition of John domains. Let $z \in A_\infty$ and let Γ be the geodesic on $A_\infty \cup \{\infty\}$ which contains z and ∞. Here we mean geodesic in the Poincaré metric $\rho(\cdot,\cdot)$ on $A_\infty \cup \{\infty\}$. Then the point z splits Γ into two arcs. We let γ_z be that arc which does *not* contain ∞. In other words, γ_z is a (half) geodesic which runs from z "to J". Then

$$P(\gamma_z) = \gamma_{P(z)}, z \in A_\infty,$$

and

$$G(P(z)) = dG(z),$$

where $G(z)$ is Green's function for A_∞ with pole at ∞, and d is the degree of P.

Lemma 3.2. *There are $C, \alpha > 0$ such that whenever $\delta(z) \leq 1$,*

$$\delta(z) \leq CG(z)^\alpha, \quad z \in A_\infty.$$

Proof. Fix $\varepsilon > 0$ such that condition (B) of Theorem 2.1 holds and suppose $\delta(z_0) < \varepsilon$. Then if $P^n(z) = z_0$, $\delta(z) \leq C\theta^n$ and $G(z) = d^{-n}G(z_0)$. □

Lemma 3.3. $\text{length}(\gamma_z) = \ell(\gamma_z) \leq CG(z)^\alpha$, $\delta(z) \leq 1$.

Proof. Let $z = z_0$ and pick $z_n \in \gamma_z, n \geq 1$, such that

$$\rho(z_{n-1}, z_n) = 1, n \geq 1.$$

Then by Koebe,

$$\ell(\gamma_z) \sim \sum_{n=0}^{\infty} \delta(z_n)$$

and since $G(z_n) \sim \lambda^n G(z_0)$ for some $\lambda < 1$, it follows from Lemma 3.2 that

$$\ell(\gamma_z) \leq C \sum_{n=0}^{\infty} \lambda^{\alpha n} G(z_0)^{\alpha}.$$

We now use a condition that is equivalent to the John condition, for simply connected domains, namely, there is $M > 0$ such that whenever $z \in A_\infty$, $\delta(z) \leq 1$, then $w \in \gamma_z$ and $\rho(w, z) \geq M$ implies

$$\delta(w) \leq \frac{1}{2}\delta(z). \tag{3.1}$$

That (3.1) is equivalent to the standard definition is not difficult (see e.g. [**J**]).

Lemma 3.4. *Condition (3.1) holds for A_∞.*

Proof. Let ε be as in condition (B) of Proposition 2.1. Then by Lemma 3.3 there is $\delta > 0$ such that whenever $G(z) \leq \delta$, there is $x \in J$ such that

$$\gamma_z \subset B(x, \frac{\varepsilon}{2}). \tag{3.2}$$

By compactness, there is for every $\eta > 0$ an $M > 0$ such that whenever $G(z) \sim \delta$ and $z' \in \gamma_z, \rho(z, z') \geq M$, then

$$\delta(z') \leq \eta\delta(z). \tag{3.3}$$

Now let $w \in A_\infty, G(w) \leq \delta$, and assume n is the integer such that

$$d^{-n-1}\delta < G(w) \leq d^{-n}\delta.$$

Applying P^{-n} on $B(x, \varepsilon)$ (with x as in condition (3.3) for $z = P^n(w)$) along with Lemma 2.2, we see that if $\eta > 0$ is small enough there is M such that for $w' \in \gamma_w$ and $\rho(w, w') \geq M$,

$$\delta(w') \leq \frac{1}{2}\delta(w),$$

as was demanded.

Case II. A_∞ is infinitely connected. The proof requires only minor modifications from Case I. We must replace geodesics by Green's lines. Recall that a Green's lines for $G(z)$ (with pole at ∞) is a curve of steepest descent for G. These lines may bifurcate at a critical point of G so that we choose γ_z to be a Jordan arc, which is a subarc of the Green's line through z, and which "runs to J". (In other words, if z lies on a critical Green's line, we choose a direction at every critical point encountered. The choice of direction, left or right, must be taken consistently.) Then $\gamma_{P(z)} = P(\gamma_z)$ so we can proceed. It is useful later in this paper to record that the hyperbolic metric behaves as in the simply connected case, i.e.

$$d\rho(z) \sim \delta(z)^{-1}|dz|, \quad \delta(z) \leq 1. \tag{3.4}$$

To prove (3.4), notice that the estimate $d\rho(z) \leq C\delta(z)^{-1}|dz|$ is true for any planar domain. To obtain the opposite conclusion, let $\alpha = \min\{G(z) : P'(z) = 0\}$ so that $\alpha > 0$. Suppose $d^{-2}\alpha < G(z) \leq d^{-1}\alpha$. Then there is $\varepsilon > 0$ such that every branch P^{-n} on $D = \{z' : |z-z'| \leq \varepsilon\}$ is univalent, because there are no critical values of P^n on D. Here we pick $\varepsilon > 0$ so that distance $(D, J) \geq \varepsilon$. Let $w = P^{-n}(z)$ and $D_n = P^{-n}(D)$. Then

$$\sup_{w' \in D_n} \frac{G(w')}{G(w)} = \sup_{z' \in D} \frac{G(z')}{G(z)} \geq 1 + \eta$$

for some $\eta > 0$ independent of z. Therefore condition (3.4) holds, and J is said to be *uniformly perfect* [**P**]. This concludes the proof of (1.1) $\Rightarrow$ (1.3).

Remark 1. We have actually shown that A_∞ is a John domain if and only if every Green's line γ terminates at some point $z_0 \in J$ and for all $z \in \gamma$,

$$\delta(z) \geq \varepsilon|z - z_0|.$$

The proof of the implication (1.1) $\Rightarrow$ (1.4) needs only a bit more work. By Sulllivan's non wandering theorem every component $\mathcal{F}_j$ is preperiodic, so it is sufficient to treat the case where $\mathcal{F}_j = \mathcal{F}$ is periodic. By taking an iterate if necessary, we may assume $P(\mathcal{F}) = \mathcal{F}$. Now since $\partial\mathcal{F}$ is locally connected, the maximum principle implies $\partial\mathcal{F}$ is a Jordan curve. Fix $z_0, z_1 \in \partial\mathcal{F}$ and let γ be the arc of smaller diameter between z_0 and z_1, $\gamma \subset \partial\mathcal{F}$. Let n be the last integer such that $\text{diameter}(P_n(\gamma)) = \text{diameter}(\gamma^n) \leq \frac{1}{2}$ diameter$(\partial\mathcal{F})$, and set $w_j = P_n(z_j), j = 0, 1$. Then by compactness,

$$\text{diameter}(\partial\mathcal{F}) \sim |w_0 - w_1|\cdot$$

and hence by Lemma 2.2,

$$\text{diameter}(\gamma) \sim |z_0 - z_1|.$$

In other words, $\partial\mathcal{F}$ is a quasicircle.

Remark 2. With just a bit more work one sees that there is $M < \infty$ such that every $\partial\mathcal{F}_j$ is an M-quasicircle.

4. John Implies Semi-Hyperbolicity - Step I

In this section, we complete the proof of Theorem 1.1 under the additional assumption that A_∞ is *simply connected.* The infinitely connected case is discussed in Section 5, where some technical modifications are needed. The basic idea is to show that for all $x \in J$ and $r > 0$ there is a point $z_0 \in A_\infty \cap B(x,r)$ such that $\delta(z_0) \geq cr$ and $G(z) \leq CG(z_0)$ in, say, $B(x, 2r)$. If $G(z_0) \sim d^{-n}$, these two conditions will imply that P^n has the correct properties of a scaling function.

Fix $r > 0, x \in J$, and a disk $B(x,r)$. We will show that P^n is of bounded degree on $B(x,r)$ for n suitably large, i.e. part (C) of Proposition 2.1 holds. In [**J**], Section 4, the construction shows that for some constant $M < \infty$, independent of r, we can cover $A_\infty \cap \{\delta(z) < r\}$ by simply connected (ε') John domains Ω_j having the following properties

(4.1) $\text{diameter}(\Omega_j) \leq Mr$

(4.2) Ω_j and Ω_k have disjoint interiors when $j \neq k$.

(4.3) There is a point $z_j \in \Omega_j$ with $\delta(z_j) \sim r$ and

$$\sup_{z\in\Omega_j} G(z) \leq MG(z_j).$$

Here $G(z)$ is Green's function for A_∞ with pole at ∞.

By (4.1)-(4.3) there are N domain Ω_j, where N is bounded by a constant independent of x, r, such that

$$A_\infty \cap B(x, 2r) \subset \bigcup_{j=1}^{N} \Omega_j$$

and (for some other constant $\varepsilon > 0$) there is t, $2r \leq t \leq 3r$ such that either Ω_j is inside $\{|x - z| \leq t\}$ or

$$\Omega_j \cap \{|x - z| = t \pm t'\} \neq \phi, \text{ for all } t',\ 0 \leq t' \leq \varepsilon r. \tag{4.4}$$

Let Ω be the polynomial hull of $B(x,t) \cup \bigcup \Omega_j$, so that Ω is simply connected and let $\rho(\cdot,\cdot)$ be the Poincaré metric on Ω. Then by construction,

$$\rho(z, z_j) \leq C_0, \quad |z| \leq r, \quad 1 \leq j \leq N. \tag{4.5}$$

Suppose $G(z_1) \geq G(z_j)$, $1 \leq j \leq N$ and let $G(z_1) \sim d^{-n}$. Then by (4.3), $|P^n(z)| \leq C_0, z \in \Omega$, because $|P^n(z)| \leq C, z \in K, n \geq 0$. On the other hand, the distortion theorem for univalent functions (or just Koebe) shows

$$|\frac{d}{dz}P^n(z_1)|\delta(z_1) \sim \delta(P^n(z_1)) \sim 1$$

and by the first estimate in (4.3), ($\delta(z_1) \sim r^{-1}$),

$$|\frac{d}{dz}P^n(z_1)| \sim r^{-1}. \tag{4.6}$$

Condition (C) of Proposition 2.1 now follows from (4.5), (4.6) and the following easy remark:

Suppose F is holomorphic on $\mathbb{D}$, $|F(z)| \leq 1$, and there is $z_1 \in \mathbb{D}$ with $\rho(0, z_1) \leq M$ and $|F'(z_1)| \geq \varepsilon$. Then F is of degree $\leq K$ on $\{|z| \leq r\}$ where $K = K(M, \varepsilon, r)$ and $r < 1$.

5. The Infinitely Connected Case

In this section, we conclude the proof of Theorem 1.1, by showing that if A_∞ is an infinitely connected (ε) John domain, P is semi-hyperbolic. An examination of the argument at the end of Section 4 shows that it is sufficient to prove that there exist $\varepsilon_1 > 0$ and $C < \infty$, depending only on ε, such that for every $x \in J$ and $r <$ diameter(J), there is $A, 1 \leq A \leq C$, and $z_0 \in B(x, Ar)$ such that

$$\delta(z_0) \geq \varepsilon_1 r \text{ and } \sup_{z \in B(x,Ar)} G(z) \leq CG(z_0). \tag{5.1}$$

We first require some information on Green's lines. Let γ be a Green's line, $z_1, z_2 \in \gamma$, $G(z_1) \leq G(z_2)$, and suppose $\rho(z_1, z_2) \geq 1$.

Then if $\delta(z_1) \leq 1$, it follows from (3.4) and the paragraph following it, that

$$G(z_1) \leq \theta G(z_2),$$

where $\theta < 1$ is independent of z_1, z_2. This implies there is $c > 0$ such that if $\gamma(z_1, z_2)$ is the subarc of γ connecting z_1 to z_2, and $\rho(z_1, z_2) \geq 1$,

$$G(z_2) \geq G(z_1) \exp\left\{ c \int_{\gamma(z_1,z_2)} \frac{ds(w)}{\gamma(w)} \right\} \tag{5.2}$$

where ds denotes the element of arclength. This estimate is a weak version of the Ahlfors estimate for simply connected domains, see e.g. [**B**] and [**C,J**].

Now let $z_1, z_2 \in \gamma, |z_1 - z_2| \leq r$. We will divide $\gamma(z_1, z_2)$ into a bounded number of equivalence classes. Let $\{w_1, \ldots, w_N\}$ be a collection of points in A_∞ such that distance $(w_j, \gamma(z_1, z_2)) \leq r, 1 \leq j \leq N$, such that

$$\delta(w_j) \geq \varepsilon r, \quad 1 \leq j \leq N,$$

and such that whenever $w \in A_\infty$ and $\delta(w) \geq \varepsilon r$, distance $(w, \gamma(z_1, z_2)) \leq r$,

$$\inf_{1 \leq j \leq N} \rho(w, w_j) \leq 1. \tag{5.3}$$

By the (ε) John condition, we can find such a collection with

$$N \leq \text{ Const. } \varepsilon^{-2}.$$

For each $z \in \gamma(z_1, z_2)$ we fix an index j such that the John curve Γ from z to ∞ contains a point w with $\rho(w, w_j) \leq 1$. We say that $z \in S_j = S_j(\gamma(z_1, z_2))$ and notice by the John condition,

$$\rho(z, w_j) \leq C \log \frac{r}{\delta(z)}.$$

It follows from Harnack, (5.3), and (3.4) that

$$\left| \log \frac{G(z)}{G(w_j)} \right| \leq C' \log \frac{r}{\delta(z)}.$$

This means that if $z, z' \in S_j$ and $G(z) \leq G(z')$, then there is a constant $A = 2C'$ such that

$$G(z') \leq G(z) \exp\{A \log \frac{r^2}{\delta(z)\delta(z')}\}. \tag{5.4}$$

Lemma 5.1. *Suppose $f(x) > 0$ is a measurable function, integrable over $[0,1]$, and that $[0,1]$ is the disjoint union of N measurable sets $\tilde{S}_1, \ldots, \tilde{S}_N$. Suppose further that whenever $x, x' \in \tilde{S}_j$, $x < x'$,*

$$\int_x^{x'} \frac{dt}{f(t)} \leq A \log \frac{1}{f(x)f(x')}.$$

Then $\sup\limits_{x\in[0,1]} f(x) \geq c(N, A) > 0.$

Proof. By renumbering, we may assume $|\tilde{S}_1| \geq N^{-1}$. Define $F_j = \{x \in \tilde{S}_1 : 2^{-j} \leq f(x) < 2^{-j+1}\}$. Then since we may as well assume $F_j = \phi$ when $j \leq 0$, there is $k \geq 1$ such that

$$|F_k| \geq 6\pi^{-2}N^{-1}k^{-2}.$$

Let $x_1 = \inf\limits_{F_k} x, x_2 = \sup\limits_{F_k} x$. Then

$$\int_{x_1}^{x_2} \frac{dt}{f(t)} \geq \int_{F_k} \frac{dt}{f(t)} \geq 2^{k-1}|F_k|,$$

while by the hypothesis on $\tilde{S}_1$,

$$\begin{aligned}\int_{x_1}^{x_2} \frac{dt}{f(t)} &\leq A \log \frac{1}{f(x_1)f(x_2)} \\ &\leq A\ k \log 4.\end{aligned}$$

Combining the last three inequalities yields

$$6\pi^{-2}N^{-1}k^{-2}2^{k-1} \leq A\ k \log 4,$$

i.e. $\sup\limits_{x\in[0,1]} f(x) \geq 2^{-k}$ and $k \leq k_0(A, N)$. ■

We now return to our Green's line $\gamma(z_1, z_2), |z_1 - z_2| \leq r$, which is divided into N disjoint classes S_j. By (5.2) and (5.4), whenever

$z, z' \in S_j$,

$$\int\limits_{\gamma(z,z')} \frac{ds(w)}{\delta(w)} \le c^{-1} A \ \log \frac{r^2}{\delta(z)\delta(z')}. \tag{5.5}$$

If the arclength of $\gamma(z_1, z_2) = r$, we may pull back (5.5) to $[0,1]$. (Let $\varphi : [0,1] \to \gamma(z_1, z_2), |\varphi'| \equiv r$, and set $f(x) = r^{-1}\delta(\varphi(x))$.) We obtain by Lemma 5.1,

$$\sup_{z \in \gamma(z_1, z_2)} \delta(z) \ge \eta\, r, \tag{5.6}$$

where $\eta > 0$ depends only on the John constant.

We now use (5.6) to prove (5.1). Fix $x \in J$ and $r > 0$. Let $C_1 \gg 1$ be a large constant, and suppose that for $j \in \mathbf{N}, 2 \le j \le M$,

$$\sup_{\substack{z \in B(x, jr) \\ \delta(z) \ge C_1^{-1} r}} C_1 G(z) \le \sup_{z \in B(x, jr)} G(z). \tag{5.7}$$

Assuming $M \gg \varepsilon^{-2}$, we will draw a contradiction as follows.

Let $G(z)$ assume its maximum over $B(x, jr)$ at the point w_j. Let γ_j be a Green's line passing through $w_j, \gamma_j = \gamma(w_j, \tilde{w}_j)$, $G(w_j) < G(\tilde{w}_j)$, where length$(\gamma_j) = r$. Then by (5.6), there is $z_j \in \gamma_j$ with $G(z_j) \ge G(w_j)$ and

$$\delta(z_j) \ge \eta r. \tag{5.8}$$

Now if $M = \text{Const.}\, \varepsilon^{-2}$, there are two indices $j < k \le M$ such that there exist John curves Γ_j and Γ_k (from z_j, z_k) and a point $w \in B(x, 2Mr)$ such that

$$\rho(w, \Gamma_j), \rho(w, \Gamma_k) \le 1.$$

From (5.8), it follows as in the argument for (5.4), that

$$\begin{aligned} G(z_k) &\le G(z_j) \exp\left\{A \ \log \frac{4M^2 r^2}{\delta(z_j)\delta(z_k)}\right\} \\ &\le (4M^2\eta^{-2})^A G(z_j). \end{aligned} \tag{5.9}$$

Now by (5.7), $C_1 G(z_j) \le G(z_{j+1}) \le G(z_k)$. (Here we use $z_j \in B(x, (j+1)r)$ and $G(z_j) \ge G(w_j)$.) But by (5.9) this implies

$$C_1 \le (4M^2\eta^{-2})^A,$$

which is a contradiction. Therefore, if $C_1 > (4M^2\eta^{-2})^A$, (5.7) fails for some $j \le M$, i.e. there exists $\tilde{z} \in B(x, jr)$ such that $\delta(\tilde{z}) \ge C_1^{-1} r$ and

$$\sup_{z \in B(x,jr)} G(\tilde{z}) \le C_1 G(\tilde{z}).$$

Moving from $\tilde{z}$ to x by distance $\frac{1}{2}C_1^{-1}r$, we obtain a point $z' \in B(x, jr)$ with

$$\delta(z') \ge \frac{1}{2} C_1^{-1} r,$$

and by Harnack, $G(\tilde{z}) \le$ Const. $G(z')$. This means that (5.1) holds, and the proof of Theorem 1.1 is complete.

6. Flowers

Suppose $P(z)$ has a parabolic periodic point (of some order) at some point z_0. Then by conjugating and taking a high enough iterate, we may assume $z_0 = 0$ and $P'(0) = 1$, i.e.

$$P(z) = z - z^N + \cdots . \tag{6.1}$$

Then P has $N - 1$ "flower petals" $\mathcal{F}_1, \ldots, \mathcal{F}_{N-1}$ meeting at 0, and $P(\mathcal{F}_j) = \mathcal{F}_j, 1 \le j \le N - 1$. We will make a natural assumption on the critical orbit:

If $P'(c) = 0$, either all but finitely many iterates $P^n(c) \in \mathcal{F}_j$ for some j, or $\omega(c) \cap \partial\mathcal{F}_j = \phi$ for all j. (6.2)

This means that either $P^n(c) \in \mathcal{F}_j$ for some j, or $\omega(c)$ misses a neighborhood of $\cup\mathcal{F}_j$.

Remark. When P is an iterate of P_0 and P_0 has only one critical point, assumption (6.2) holds automatically because each $\mathcal{F}_j$ contains a critical point of P.

The main reason we require condition (6.2) is that by modifying the reasoning of Douady and Hubbard [**D,H**], it implies

$$\partial\mathcal{F}_j \text{ is locally connected.} \tag{6.3}$$

We will prove (6.3) in Appendix 1.

Theorem 6.1. *Under hypotheses (6.1) and (6.2) we have the following dichotomy:*

(A) *If $N = 2$, $\partial\mathcal{F}_1$ is not a quasicircle, but $\mathcal{F}_1$ is a John domain.*

(B) *If $N \geq 3$, i.e. if there are at least two petals, then $\partial\mathcal{F}_j$ is a quasicircle, $1 \leq j \leq N-1$ (and hence $\mathcal{F}_j$ is a John domain).*

Proof. We require some classical facts about flower petals. By renumbering, we have for some $\varepsilon > 0$ that

$$(0, \varepsilon e^{i\frac{2\pi(j-1)}{N-1}}] \subset \mathcal{F}_j.$$

We may as well choose $\mathcal{F}_j = \mathcal{F}_1 \equiv \mathcal{F}$. Then for every $\delta > 0$ there is $R > 0$ such that

$$W \equiv \{re^{i\theta} : 0 < r < R, |\theta| < \frac{\pi}{N-1} - \delta\} \subset \mathcal{F} \tag{6.4}$$

and

$$\mathcal{F} \cap \{|z| < R\} \subset \{re^{i\theta} : 0 < r < R, |\theta| < \frac{\pi}{N-1} + \delta\}.$$

In other words, $\partial\mathcal{F}$ is well approximated at the origin by an angle. Note that this also implies, if $N = 2$, that $\partial\mathcal{F}$ is not a quasicircle.

Since P is a polynomial, $\mathcal{F}$ is simply connected, and we may conjugate P to a Blaschke product $B(z)$ on $\mathbb{D}$, with a parabolic fixpoint at 1. Then degree $(B) \geq 2$ because P has a critical point in $\mathcal{F}$. Any such Blaschke product has Julia set $= \mathbf{T}$, i.e., it has two flower petals $\mathbb{D}$ and $\mathbb{D}^*$. Incidentally, this means that if we put $w = z - 1$, then in the w coordinate system B has form

$$B(w) = w - aw^3 + \cdots$$

for some $a > 0$.

The critical orbits $\omega(c)$ for B lie in a Stolz cone which is either in $\mathbb{D}$ or $\mathbb{D}^*$ according to where c lies, and $\omega(c)$ accumulates only at 1. Because of this, the reasoning of Douady and Hubbard applies and there is a C^∞ metric $\tilde{\lambda}(z)|dz|$ such that

$$\tilde{\lambda}(z) \equiv 1 \text{ for } z \text{ near } 1, |z-1| \leq \alpha$$

and B is hyperbolic in the $\tilde{\lambda}$ metric near $\mathbf{T}$ and away from 1:

$$|D_{\tilde{\lambda}}P(z)| \equiv \frac{|P'(z)|\tilde{\lambda}(P(z))}{\tilde{\lambda}(z)} \geq 1+\beta > 1, \tag{6.5}$$
$$1-\beta < |z| < 1+\beta, |1-z| \geq \alpha.$$

See [**C,G**] or [**D,H**] for details.

Mimicking the definition of (6.4) we now set

$$\tilde{W} = \{1-w : 0 < |w| < R, |\arg\ w| < \frac{\pi}{2} - \delta\}$$

so that $\tilde{W} \subset \mathbb{D}$. Then W and $\tilde{W}$ essentially map to each other under the conjugating maps. If R is small enough and c is a critical point of P with $\omega(c) \cap \mathcal{F} \neq \phi$,

$$\omega(c) \cap \{|z| < R\} \subset \{|\arg\ z| < \frac{\pi}{2N-2}\}. \tag{6.6}$$

This follows from the local dynamics of P. Similarly, if c is a critical point of B and R is small enough,

$$\omega(c) \cap \{|1-z| < R\} \subset \{|\arg(z-1)| < \frac{\pi}{2}\} \cup \{|\arg(1-a)| < \frac{\pi}{4}\}. \tag{6.7}$$

In Section 3, we used the fact that in A_∞, geodesics are mapped by P to geodesics. In our setting, this is no longer true, i.e. $B(z)$ does not map radial line segments to line segments. On the other hand, this is essentially true as long as we do not iterate too long.

We first implement this philosophy on $\mathbb{D}$. Let $z \in \mathbb{D}$ and let n be the largest integer such that

$$1-|B^k(z)| < \eta, \quad B^k(z) \notin \tilde{W}, \quad 0 \leq k \leq n.$$

Let γ_z denote the "half geodesic through z", i.e. $\gamma_z \equiv \mathbb{D} \cap \{tz : t \geq 1\}$. Then γ_z terminates at $e^{i\theta_0}$, $\theta_0 = arg\ z$ and we let $e^{i\theta_n} = B^n(e^{i\theta_0})$. Let $w = |B^n(z)|e^{i\theta_n}$ and let $\gamma^n \equiv \gamma_w$. We say $\tilde{\rho}(B^n(\gamma_z);\gamma^n) \leq M$ if whenever $z' \in \gamma_z$ and $w' \in \gamma^n$ satisfy

$$\rho(z,z') = \rho(w,w'),$$

then

$$\rho(B^n(z'),w') \leq M.$$

Here $\rho(\cdot,\cdot)$ denotes the Poincaré metric on $\mathbb{D}$.

Lemma 6.2. $\tilde{\rho}(B^n(\gamma_z), \tilde{\gamma}) \leq C_0$.

Proof. By the definition of n and (6.4) when we set $1 - |B^n(z)| = r$, there is a large constant M such that

$$B^k(c) \notin \{\xi : |\xi - B^n(z)| \leq Mr\}$$

for all $k \leq 0$ and all critical points c of B lying on the Riemann sphere. The lemma now follows from the classical distortion theorems for univalent maps (B^{-k} is univalent on the above defined disk about $B^n(z)$) and the fact that $B^n : \mathbf{T} \to \mathbf{T}$. □

We now verify that $\mathcal{F}$ is a John domain, using definition (3.1) for "John". Define as in (6.4)

$$W_0 = W \cap \{|\arg\ z| < \frac{\pi}{N-1}\}.$$

Let $z \in \mathcal{F}, \delta(z) \leq \eta$, and let n be the largest integer such that

$$\delta(P^k(z)) \leq \eta, \quad P^{k-1}(z) \notin W_0, 0 \leq k \leq n.$$

Case 1. $P^n(z) = z_n \in W_0$. Then if $\Phi : \mathcal{F} \to \mathbb{D}$ is the map conjugating P to B and if we apply Lemma 6.2, we see the geodesic (in $\mathcal{F}$) γ_z is not badly distorted by P^n. Let $\hat{\gamma} = \Phi^{-1}(\tilde{\gamma})$ where $\tilde{\gamma}$ is defined as in Lemma 6.2 with respect to the point $\Phi(z)$. Call the endpoint of $\hat{\gamma}$ by z' so that $\rho(z_n, z') \leq C_0$. Then by (6.4) we can take $M \gg 1$ (if $\delta \ll 1$) so that when $w \in \hat{\gamma}$, $\rho(w, z') = M$, we have

$$\delta(w) \ll \delta(z') \sim \delta(z_n). \tag{6.8}$$

(This implies (3.4) if $n = 0$).

Now $P^{-1}(z_n) = z_{n-1} \notin W_0$ and

$$\delta(P^{-1}(w)) \ll \delta(z_{n-1}) \equiv \delta_{n-1} \tag{6.9}$$

because taking one inverse image does not effect (6.8). Now by (6.6) there is a simply connected domain $\mathcal{D}$ with $\gamma^{n-1} = B^{n-1}(\gamma_z) \subset \mathcal{D}$,

$$\text{distance}(\gamma^{n-1}, \partial\mathcal{D}) \sim \delta_{n-1},$$

and

$$\omega(c) \cap \mathcal{D} = \phi$$

for all critical points c of P. Then P^{-n+1} is univalent on $\mathcal{D}$, so by the distortion theorem and (6.9),

$$\delta(P^{-n}(w)) \ll \delta(z),$$

i.e. (3.1) holds.

Case 2. $P^n(z) = z_n \notin W_0$. Then $\delta(z_n) \geq \eta$ and, since $\delta(z_{n-1}) < \eta$, it must be that $\delta(z_n) \leq C\eta$. Now since $\partial\mathcal{F}$ is a Jordan curve, γ_{z_n} cannot terminate at 0. (If γ_z terminates at 0 and $\delta(z) \leq C\eta \ll R$, then by elementary arguments - e.g. normal families - $z \in W_0$.) To be a bit more precise, let z' and $\hat{\gamma}$ be as in Case 1. Then

$$\text{distance}(\hat{\gamma}, W_0) \geq \alpha > 0$$

and $\hat{\gamma}$ also has distance $\geq \alpha$ to all critical orbits $\omega(c)$. Consequently, there is a simply connected domain $\mathcal{D}$ such that $\hat{\gamma} \subset \mathcal{D}, \omega(c) \cap \mathcal{D} = \phi$ when $P'(c) = 0$, and

$$\text{distance}(\hat{\gamma}, \partial\mathcal{D}) \geq \beta > 0.$$

Now since $\partial\mathcal{F}$ is a Jordan curve, there is $M < \infty$ such that for $w \in \hat{\gamma}$ and $\rho(w, z') = M$ (where $\rho(\cdot,\cdot)$ is on $\mathcal{F}$),

$$\delta(w) \ll \delta(z').$$

This follows from a compactness argument and the fact that $\delta(z') \geq c\eta$. Since P^{-n} is univalent,

$$\delta(P^{-n}(w)) \ll \delta(z),$$

and our result now follows from Lemma 6.2.

We have established for Part (A) of the theorem that $\partial\mathcal{F}$ is a John domain. When there is only one petal, $\mathcal{F}$ contains all $z \neq 0, |\arg\ z| < \pi - \delta$ when $|z|$ is small. Thus $\partial\mathcal{F}$ has a cusp at 0 and is not a quasicircle. This completes the proof of Part (A).

We now turn to the proof of Part (B). Our first step is to find the correct definition of quasicircles, and this turns out to be the following criterion. Let Γ be a Jordan curve bounding a bounded domain Ω. Then Γ is a quasicircle if and only if Ω is a John domain and for all points

$w, z \in \Omega$, the Poincaré geodesic γ from w to z satisfies

$$\text{diameter}(\gamma) \leq M|w - z|. \tag{6.10}$$

Since $\partial\mathcal{F}$ is a John curve, a compactness argument shows it is sufficient to treat the case where $|w - z|$ is small. Let $\gamma^n = P^n(\gamma)$ and let $\xi_n \in \gamma^n$ maximize $\delta(\xi)$. We choose n to be the first integer so that either $\gamma^n \cap W_0 \neq \phi$ or $\delta(\xi_n) \geq \eta$.

Case 1. $\gamma^n \cap W_0 \neq \phi$. Let $w_n = P^n(w), z_n = P^n(z)$. Then by (6.4), and the fact that $\mathcal{F}$ is a John domain, it is easy to see that $\text{diameter}(\gamma^n) \leq M|w_n - z_n|$ and ξ_n is within a bounded hyperbolic distance (on $\mathcal{F}$) of W_0. Pulling back $P^{-1}(\gamma^n) = \gamma^{n-1}$ we still have $\text{diameter}(\gamma^{n-1}) \leq M|w_{n-1} - z_{n-1}|$. Now since $\gamma^{n-1} \cap W_0 = \phi$, it follows from (6.4) that

$$\text{distance}(\gamma^{n-1}, \omega(c)) \geq \alpha > 0$$

for all critical c. As in the proof of Part (A), we build a domain $\mathcal{D}$ such that $\gamma^{n-1} \subset \mathcal{D}$, distance $(\gamma^{n-1}, \partial\mathcal{D}) \geq \beta$, and P^{-n+1} is univalent on $\mathcal{D}$. The result now follows from the distortion theorem for univalent functions.

Case 2. $\delta(\xi_n) \geq \eta, \gamma^n \cap W_0 = \phi$. Then by (6.4) $|z_n|, |w_n| \geq \varepsilon_0 |z_n - w_n|$, so there is again a good domain $\mathcal{D}$ containing γ_n on which P^{-n} is univalent.

7. The Periodic Table

In this section we discuss the table for $z^2 + c$ presented in the introduction. Our results follow easily from the previous sections plus works of Herman [**H**] and Mañé [**Ma**].

Case 1. $c \notin M =$ Mandelbrot set. Then J is totally disconnected and it is well known that P is hyperbolic on J. To see this, simply note that if $\varepsilon =$ distance$(0, J)$, P^{-n} is univalent on $B(x, \delta), x \in J$. Now A_∞ is John by Theorem 1.1.

Case 2. Attractive Basin. This case is proved in [**C,J**]. It is the same as Case 1 because distance$(\omega(0), J) > 0$, so Theorem 1.1 applies.

Case 3. Parabolic Basin, One Petal. This is Theorem 6.1, Part (A).

Case 4. Parabolic Basin, ≥ 2 Petals. This is Theorem 6.1, Part (B).

Case 5. Siegel disk, $\alpha \in C.T.$ By a remarkable results due to Herman [**H**], the Siegel disk is bounded by a quasicircle, and hence so is every bounded $\mathcal{F}_j$. A_∞ cannot be John because condition (D) of Theorem 6.1 fails for x in the boundary of the Siegel disk.

Case 6. Siegel disk, $\alpha \in D \setminus C.T.$ The results of [**H**] assert for certain values of α, the boundary of the Siegel disk is a Jordan curve, but not a quasicircle. (Whether the boundary is always a Jordan curve for $\alpha \in D$ is not yet known.) As in Case 5, A_∞ is not a John domain.

Case 7. Siegel disk, $\alpha \in B \setminus D$. We do not know much about this case, except that Michael Herman [**H**] has proved the existence of $\alpha \in B \setminus D$ with a Siegel disk bounded by a quasicircle.

Case 8. Dendrite, $0 \notin \omega(0)$. Then by Theorem 1.1 (Estimate (1.2)), A_∞ is John.

Case 9. Dendrite, $0 \in \omega(0)$. Then by Lemma 2.3, A_∞ is not John.

8. An Example

In this section we show by pictures how J is built in the case $z^2 + c, c \in \mathbb{R} \cap M$, $0 \notin w(0)$, and we can "see" exactly why A_∞ is John. In this case, the Julia set intersects $\mathbb{R}$ in an interval $I = [a, b]$ (assuming there are no bounded components $\mathcal{F}_j$) and a, b are given by

$$\frac{1 \pm \sqrt{1-4c}}{2}.$$

Since c < 0, we have the interval I_0 from 0 to id in J, where $c - d^2 = a$. The Julia set is now given by the closure of

$$I \cup \{P^{-1}(I_0) : n \geq 0\},$$

and we notice that by Proposition 2.1, diameter$(P^{-n}(I_0)) \leq C\theta^n$ so that these "intervals" are shrinking geometrically. Suppose we did not know Theorem 1.1, and attempted to visually inspect J to determine whether A_∞ is John.

The first obstruction to "Johnness" would be that an interval $I_j^n = P^{-n}(I_0)$ has center $x_j^n \in \mathbb{R}$ and I_j^n makes a small angle with $\mathbb{R}$ at x_j^n. (More precisely, $I_j^n \cap \mathbb{R}_+^2 \not\subset \{|\frac{\pi}{2} - \arg(z - x_j^n)| < \frac{\pi}{2} - \varepsilon\}$.

A look at the picture in Figure 1 shows that the I_j^n start bending as n increases, but the bending stabilizes so as to prevent small angles. (One could observe that this easily follows from the distortion theorem for univalent functions and the hypothesis $0 \notin \omega(0)$.)

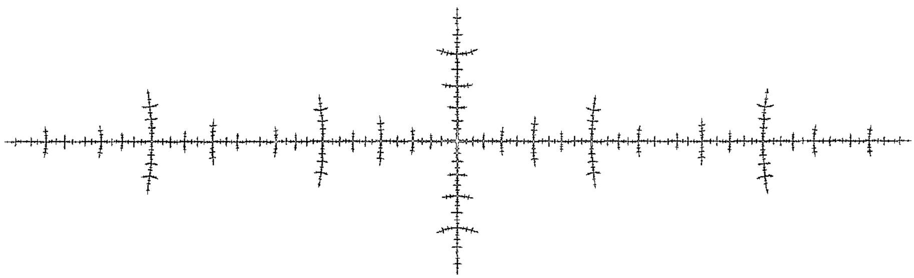

Figure 1

Let $\mathcal{F}_1 = \{I_j^n : I_j^n \pitchfork \mathbb{R}\}$ where $\pitchfork$ means transverse, and define inductively

$$\mathcal{F}_k = \{I_j^n : I_j^n \pitchfork I' \text{ for some } I' \in \mathcal{F}_{k-1}\}.$$

The same reasoning as in the previous paragraphs shows $I_j^n \in \mathcal{F}_k$ does not make a small angle with $I' \in \mathcal{F}_{k-1}$, and this is seen readily in the picture. We define in a natural way $J(I_j^n), I_j^n \in \mathcal{F}_k$, as the piece of J which is "attached" to I_j^n, but not to $I' \in \mathcal{F}_{k-1}$, where $I_j^n \pitchfork I'$. The reason that A_∞ is John is that $\mathrm{distance}(J(I_j^n), J(I_\ell^m)) \geq c \; \min(\mathrm{diameter}(J(I_j^n), \; \mathrm{diameter}(J(I_\ell^m))))$ and one also sees this in the picture below.

Appendix 1. Locally Connected Petals

In this appendix we outline a proof that, under the assumptions in Section 6, flower petals have locally connected boundaries. (Note that we are *not* proving the entire Julia set is locally connected.) The idea is to combine the ideas of Douady-Hubbard [**D,H**] with elementary properties of the petals at the parabolic fixpoint.

Let us assume that $P(z) = z - z^2 + H.O.T.$ has a parabolic fixpoint at the origin. Then P has one petal, $\mathcal{F}$, with fixpoint at 0. (The proof when P has more than one petal is virtually identical.) Our assumption is that if $P'(c) = 0$, either $\omega(c) \setminus \mathcal{F}$ is finite (i.e. some iterate of c lands in $\mathcal{F}$) or $\omega(c)$ has positive distance to $\mathcal{F}$. Following [**D,H**], we take a finite sheeted cover $\mathcal{R}$ of

$$\bar{\mathbb{C}} \setminus \omega(\mathcal{C}),$$

where $\omega(\mathcal{C})$ is the union of all $\omega(c), P'(c) = 0$. The idea of [**D,H**] is that there is a metric λ (coming from the Poincaré metric on the Riemann surface $\mathcal{R}$) such that P is *expanding* in the λ metric by a definite factor, if we stay away from $\omega(\mathcal{C})$. As in [**D,H**] we can thus modify λ to be identically 1 when $|z| \leq 2\rho, \lambda > 0$ is C^∞ in a neighborhood of $\mathcal{F} \setminus \{|z| \leq 2\rho\}$. Then

$$D_\lambda P(z) \equiv \frac{|P'(z)|\lambda(P(z))}{\lambda(z)} \geq 1 + \varepsilon \tag{9.1}$$

if $z \in \mathcal{F} \cap \{|z| > \rho\}$. (Note: ρ, not 2ρ.) By taking ε small enough, we may take ρ as small as we please. Then for all $z \in \mathcal{F}$, $D_\lambda P(z) \geq 1$ as long as we avoid the wedge

$$W = \{0 < |z| < 2\rho, |\arg\ z| \leq \frac{7}{8}\pi\}.$$

Note that $W \subset \mathcal{F}$ if ρ is small enough. On W we have the "trivial" estimate

$$G(z) \geq |z|, \text{ if } \rho \leq \rho_0, \tag{9.2}$$

for Green's function $G(z)$ with pole at some fixed $z_0 \in \mathcal{F}$. This is because $\mathcal{F}$ contains "asymptotically" $\{|\arg\ z| < \pi\}$ as we approach zero. (Actually, G looks essentially like $|z|^{\frac{1}{2}}$ in W, but we don't need a sharp estimate).

Now let A be a large positive number and suppose

$$A^{-2m-1} < G(z) \leq A^{-2m}.$$

Observe that $G(P(z)) \leq AG(z), z \in \mathcal{F}$; the existence of such an A is easily seen by moving to $\mathbb{D}$, where P is conjugate to a Blaschke product. Our aim is to prove

$$\delta_\lambda(z) \leq c_1 m^{-3/2}, \tag{9.3}$$

where $\delta_\lambda(\cdot)$ denotes the distance in the λ metric to $\partial\mathcal{F}$. This would immediately imply that $\partial\mathcal{F}$ is locally connected because if $z \in \mathcal{F}$ satisfies $G(z) \sim A^{-2n}$, then for γ_z the "half geodesic" defined in section 6, we have a collection of points z_m on γ_z with $G(z_m) \sim A^{-2m}$ and by Koebe,

$$\begin{aligned}\text{length}(\gamma_z) &\leq C \ \text{length}_\lambda(\gamma_z)\\ &\leq c' \sum_{m=n}^{\infty} \delta_\lambda(z_m)\\ &\leq c'' \sum_{m=n}^{\infty} m^{-3/2} \sim n^{-1/2}.\end{aligned}$$

As noted in (9.2), estimate (9.3) need only be verified for $z \notin W$. We therefore fix $z \in \mathcal{F}\backslash W$, with z close to $\partial\mathcal{F}$, and we suppose by induction that (9.3) holds for $1, 2, \ldots, m-1$, and $A^{-2m-1} < G(z) \leq A^{-2m}$. Because P is conjugate to a Blaschke product on $\mathbb{D}$ (see Section 6) we see (following a short argument on the image of W in $\mathbb{D}$) that there is a smallest n such that either

$$P^n(z) \in W$$

or

$$A^{-2m+2} \geq G(P^n(z)) > A^{-2m+1}. \tag{9.4}$$

We first suppose $P^n(z) \in W$ and break into different cases.

Case 1. For some k, $1 \leq k \leq n$, $|P^k(z)| \geq \rho$. Then by (9.1),

$$\begin{aligned}\delta_\lambda(z) &\leq (1+\varepsilon)^{-1}\delta_\lambda(P^k(z))\\ &\leq (1+\varepsilon)^{-1}\delta_\lambda(P^n(z))\end{aligned}$$

and by (9.2),

$$\begin{aligned}\delta_\lambda(P^n(z)) = &\ \text{distance}(P^n(z), \partial\mathcal{F})\\ &\leq |P^n(z)| \leq G(P^n(z)) \leq A^{-2m+2},\end{aligned}$$

so that (9.3) holds.

Case 2. $|P^k(z)| \leq \rho, 1 \leq k \leq n$. Then one checks by using the local behavior of P at 0 that

$$|\arg\ z| \leq (1 - A^{-3})\pi.$$

If ρ is small enough, we still have $G(z) \geq |z| \geq \delta_\lambda(z)$ so that (9.3) holds.

We now assume that $P^k(z) \notin W$, $1 \leq k \leq n$, and that (9.4) holds. Again we break into several cases.

Case 1. $|z| \geq \rho$. Then by (9.1),

$$\begin{aligned}\delta_\lambda(z) &\leq \delta_\lambda(P^n(z))(1+\varepsilon)^{-1}\\ &\leq c_1(m-1)^{-3/2}(1+\varepsilon)^{-1}\\ &\leq c_1 m^{-3/2}.\end{aligned}$$

Case 2. $1/m \leq |z| \leq \rho$. Then by (9.1) we can ignore the expansion after the first iterate and obtain, first using smoothness of P, then expansion, then induction, the estimate

$$\begin{aligned}\delta_\lambda(z) &\sim \delta_\lambda(P(z))|P'(z)|^{-1}\\ &\leq \delta_\lambda(P^n(z))|P'(z)|^{-1}\\ &\leq c_1(m-1)^{-3/2}(1+\frac{7}{4}|z|)^{-1}\\ &\leq c_1(m-1)^{-3/2}(1+\frac{7}{4m})^{-1}\\ &\leq c_1 m^{-3/2},\end{aligned}$$

because $\frac{7}{4} > \frac{3}{2}$. Here we use $\sim$ to ignore logarithmic errors of order m^{-2}

Case 3. $|z| \leq \frac{1}{m}, |P^n(z) - z| \leq C_2 m^{-2}$.

Then since the hypothesis $G(P^n(z)) \geq AG(z)$ forces the hyperbolic distance in $\mathcal{F}$ from z to $P^n(z)$ to be ≥ 1, Koebe yields

$$\begin{aligned}\delta_\lambda(z) &= \text{ distance}(z, \partial\mathcal{F})\\ &\leq cC_2|z - P^n(z)|\\ &\leq C_1 m^{-2},\end{aligned}$$

if C_1 is large enough with respect to C_2.

Case 4. $|z| \leq \frac{1}{m}, |P^n(z) - z| > C_2 m^{-2}$. We may assume $|z| \geq m^{-3/2}$.

One checks easily from the expansion

$$P(z) = z - z^2 + O(z^3)$$

that

$$n \geq cC_2 m^{-2}|z|^{-2}$$

because from step k to step $k+1$ we move by $\approx |P(z_k)|^2$. Because $P^k(z)$ is never in W for $k \leq n$,

$$\begin{aligned}|\frac{d}{dz}P^n(z)| &\geq (1+\frac{7}{4}|z|)^n \\ &\geq 1+\frac{7}{4}cC_2 m^{-2}|z|^{-1} \\ &\geq 1+\frac{7}{4}cC_2 m^{-1}.\end{aligned}$$

As in Case 3,

$$\begin{aligned}\delta_\lambda(z) &\leq \delta_\lambda(P^n(z))(1+\frac{7}{4}cC_2 m^{-1})^{-1} \\ &\leq C_1(m-1)^{-3/2}(1+\frac{7}{4}cC_2 m^{-1}) \\ &\leq C_1 m^{-3/2}\end{aligned}$$

as soon as C_2 is large enough, i.e. $\frac{7}{4}cC_2 > \frac{3}{2}$.

Notice that at several points above we could have changed m^{-1} to $m^{-\alpha}$ for $\alpha = \frac{7}{8}$, say. There is really no point because once $\mathcal{F}$ is locally connected, Section 6 shows $\mathcal{F}$ is a John domain and so $\delta(z) \leq CG(z)^\beta$. This is much stronger than (9.3).

References

[**B**] A. Beurling. *The collected works of Arne Beurling.* Birkhäuser I (1989).

[**C,G**] L. Carleson and T.W. Gamelin. *Complex dynamics.* Springer-Verlag, Universitext: Tracts in Mathematics (1993).

[**C,J**] L. Carleson and P.W. Jones. *On coefficient problems for univalent functions and conformal dimensions.* Duke Math. J. 66 (1992), 169-206.

[**D,H**] A. Douady and J. Hubbard. *Études dynamique des polynômes complexes.* Publ. Math. Orsay (1984-02, 1985-04).

[**H**] M. Herman, personal communication.

[**J**] P.W. Jones. *On removable sets for Sobolev spaces in the plane*, in Conference in honor of E.M. Stein, Princeton University Press (1993).

[**Ma**] R. Mañé. *On a lemma of Fatou.* Bol. Soc. Bras. Mat. 24 (1993), 1-12.

[**Mi**] M. Misiurewicz. *Absolutely continuous measures for certain maps of an interval.* Publ. I.H.E.S. 53 (1981), 17-52.

[N,V] R. Näkki and J. Väisälä. *John disks.* Expositiones Math. (1991), 3-43.

[P] Ch. Pommerenke. *Uniformly perfect sets and the Poincaré metric.* Arch. Math. 32 (1979), 192-199.

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